

Section 2: Modular Forms

Modular forms are very complicated objects, or perhaps a better way to put it is very complex objects. It is not easy to describe them quickly, and once described, it is not easy to understand them. Picturing them is nearly impossible. Having said that, I will attempt to describe them quickly, but I do not expect you to understand them completely, and I won't even attempt to draw them. Modular forms, roughly speaking, are functions on the complex plane that have essentially infinitely many symmetries. That's the quick version. To understand this, some history and details are warranted.

We begin with French mathematician Henri Poincaré. Poincaré was a remarkable mathematician who contributed to many different fields of mathematics. In 1895, he published *Analysis Situs*, a book on topology that essentially marks the beginning of the field. He also published books in physics, astronomy, and general science. The work of his we want to focus on involves complex analysis and non-Euclidean geometry. Poincaré studied periodic functions, much like Fourier studied sine and cosine before him. But instead of limiting himself to the real line, Poincaré studied functions $f(z)$ in the complex plane. Since the complex plane is two-dimensional, these functions have a two-dimensional domain and range and are therefore very hard to graph and picture. Poincaré considered functions like these which exhibited periodicity along both the real axis and the imaginary axis in their domains. As he studied them, he began to wonder about an even greater generalization. Being periodic is a type of symmetry, and Poincaré wondered about functions that had even more symmetries. He studied functions that remained unchanged when the complex variable z was changed according to $z \rightarrow \frac{az+b}{cz+d}$. It turns out there are infinitely many distinct transformations of this type, they all commute with each other, and the functions Poincaré was studying were invariant (unchanged) under them. Poincaré termed these functions *automorphic forms*.

Exercise 3.2.1 Consider the complex-valued function $f(z) = 5z^2 + z$. Show that $f(z)$ is NOT invariant under the transformation $z \rightarrow \frac{2z+1}{z-3}$. In other words, show that $f(z)$ and $f(\hat{z})$ (where $\hat{z} = \frac{2z+1}{z-3}$) are different functions.

The automorphic forms of Poincaré were very strange objects since they satisfied so many internal symmetries. In fact, Poincaré wasn't even sure they existed. In "Fermat's Last Theorem" by Amir Aczel, the author describes how Poincaré struggled with this idea:

For fifteen days he would wake up in the morning and sit at his desk for a couple of hours trying to convince himself that the automorphic forms he invented could not possibly exist. Day after day. Then on the fifteenth day he realized he was wrong. These strange functions, hard to imagine visually, did actually exist.

Eventually, Poincaré extended them to even more complicated functions called *modular forms*. The modular forms only live in the upper half of the complex plane and surprisingly they have hyperbolic geometry. In other words, they live in a space that is non-Euclidean. Through any point in this half-plane, many distinct lines are parallel to a given line. It's a weird place and modular functions with their infinitely many internal symmetries fit right in.

As mentioned above, complex analysis was not the only area Poincaré in which worked. His most lasting claim to fame comes in topology. Topology is essentially the study of surfaces and spaces and the continuous functions that act on them. Topologists study properties of surfaces that remain invariant under these functions. One such property is the *genus* of a surface, or how many holes it has. In three dimensional Euclidean space for example, a sphere (the surface of a ball) is a two dimensional object with no holes, so it has genus zero. A torus (doughnut shaped surface) has genus one. A hole has to run completely through the surface, so a coffee cup is also genus one while a two handled coffee cup would have genus two. (That reminds me of the old math joke: A topologist is a mathematician who cannot tell his coffee cup from his doughnut. They have the same genus.)

This brings us back to Mordell. Louis Mordell was an English mathematician who discovered a remarkable connection between solutions of algebraic equations and topology. We already saw Mordell's Theorem in Section 3.1. In essence, Mordell's Theorem was a special case of a larger conjecture by Mordell. Considering equations over the complex numbers, Mordell conjectured that if the surface of solutions had two or more holes (genus $g \geq 2$), then the equation would have only a finite number of solutions. He was unable to prove his conjecture, but in 1983 Gerd Faltings did and Mordell's Conjecture is now slowly becoming known as Faltings Theorem.

Theorem 3.2.1 (Faltings Theorem) If C is an algebraic curve of genus g with $g \geq 2$, then C only has a finite number of rational points.

Recall that Mordell's Theorem stated that if an elliptic curve had any rational points, it had infinitely many. This is a different conclusion from Faltings Theorem because elliptic curves have genus one. Neither Mordell nor Faltings had any interest in Fermat's Last Theorem, but since the genus of $x^n + y^n = z^n$ with $n > 3$ is two or more, Faltings' proof renewed the interest in FLT. If these equations had any solutions, suddenly it was only a finite number. Additionally, shortly after Faltings announced his proof, two other mathematicians showed that the number of solutions to $x^n + y^n = z^n$, if any existed at all, decreased as n approached infinity. In other words, FLT was "almost always" true. But what did modular forms and elliptic curves have to do with Fermat's Last Theorem? Or each other for that matter? We'll learn the connection in the next section.